Mathematics - Course 221

THE INTEGRAL

I The Indefinite Integral

If $F^1(x) = f(x)$, then $f(x)$ is the *derivative* of $F(x)$, and $F(x)$ is an *antiderivative* of $f(x)$. The process of finding $f(x)$ from $F(x)$ is called *differentiation*, whereas the process of finding $F(x)$ from $f(x)$ is called *integration*. Thus differentiation and integration are opposite processes:

$$F(x) \underset{\text{integration}}{\overset{\text{differentiation}}{\rightleftarrows}} f(x) = F^1(x)$$

Example 1

x^2 is an antiderivative of $2x$ since

$$\frac{d}{dx} x^2 = 2x$$

In fact, any function of the form $F(x) = x^2 + C$ is an antiderivative of $f(x) = 2x$ since

$$\frac{d}{dx}(x^2 + C) = 2x \qquad \left(\frac{d}{dx} C = 0\right)$$

"C" is called an *integration constant*.

Graphical Significance of Integration Constant

The graphical significance of the *integration constant C* is that $x^2 + C$ represents a *family of curves*, each value of C corresponding to a unique member of the family (see Figure 1). Note that every member of the family has precisely the same slope at any particular x-value, say x_1.

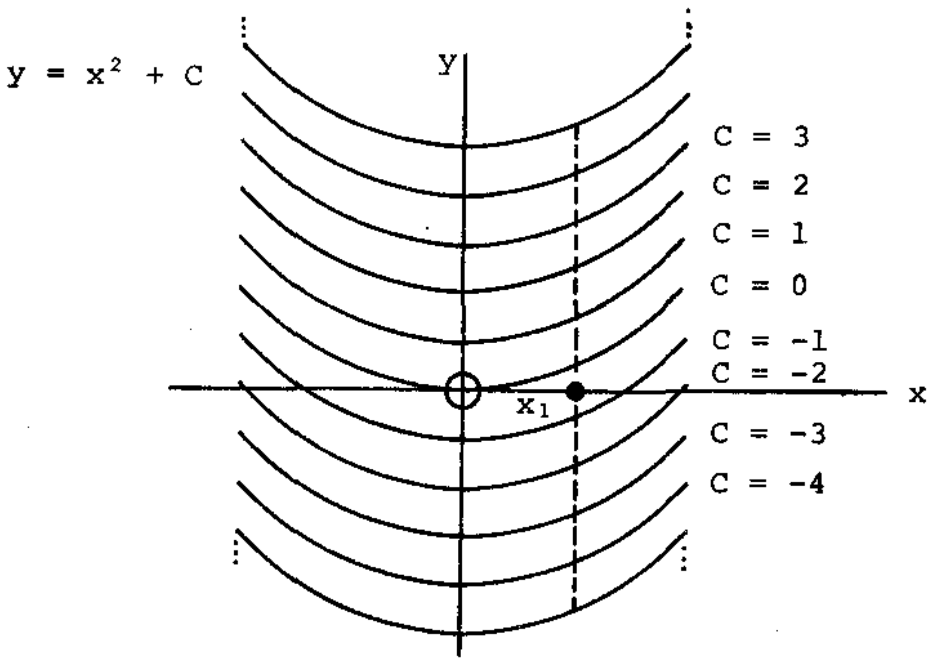

Figure 1

Boundary Condition

Integration may be regarded as the process of finding a function (curve) from its derivative (slope). As Figure 1 illustrates, there is always an infinite family of curves having the given slope.

An integration has a unique solution, however, if a *boundary condition* is imposed.

DEFINITION: A *boundary condition* is the specification of the value of the integral at a particular x-value.

The graphical significance of imposing a boundary condition is that a point is specified on the solution curve, and thus a unique curve is selected from the infinite family as the solution.

Example 2

Find the antiderivative of $f(x) = 2x$, which has the value 5 when $x = 2$.

Solution

The required antiderivative has the form

$$F(x) = x^2 + C \quad \text{(from Example 1)}$$

However, the boundary condition,

$$F(2) = 5 \Rightarrow (2)^2 + C = 5$$

$$\therefore C = 1$$

$$\therefore \quad \underline{\underline{F(x) = x^2 + 1}}$$

Note that the boundary condition in Example 2 selects the curve corresponding to C = 1 in Figure 1.

Integral Notation

"The integral of 2x with respect to x equals $x^2 + C$" is written symbolically as follows:

$$\underbrace{\int}_{\text{integral sign}} \underbrace{2x}_{\text{integrand}}\ \underbrace{dx}_{\text{differential}} = \underbrace{\underbrace{x^2}_{\text{antiderivative}} + \underbrace{C}_{\text{integration constant}}}_{\text{indefinite integral}}$$

Where:

- the *integral sign* is read "the integral of"
- the *integrand* is the function being integrated
- the *differential* "dx" indicates integration wrt x
- the antiderivative and integration constant together comprise the *indefinite integral* of 2x wrt x.

In general, the derivative of F(x) wrt x equals f(x) if and only if the integral of f(x) equals F(x) + C

ie,

$$\boxed{F^1(x) = f(x) \Longleftrightarrow \int f(x)dx = F(x) + C}$$

Example 3

$$\int 4x^3 dx = x^4 + C \quad \text{since } \frac{d}{dx}(x^4 + C) = 4x^3$$

II Displacement, Velocity and Acceleration

$$v(t) = s^1(t) \Longleftrightarrow s(t) = \int v(t)dt$$

$$a(t) = v^1(t) \Longleftrightarrow v(t) = \int a(t)dt$$

Example 4

Find v(t) and s(t) given a(t) = -10, and the boundary conditions, v(0) = 0 and s(0) = 100.

Solution

$$v(t) = \int a(t)dt$$

$$= \int -10dt$$

$$= -10t + C_1$$

But $v(0) = 0 \Rightarrow -10(0) + C_1 = 0$

$\therefore \quad C_1 = 0$

$\therefore \quad \underline{\underline{v(t) = -10t}}$

$$s(t) = \int v(t)dt$$

$$= \int -10t\ dt$$

$$= -5t^2 + C_2$$

But $s(0) = 100 \Rightarrow -5(0)^2 + C_2 = 100$

$\therefore \quad C_2 = 100$

$\therefore \quad \underline{\underline{s(t) = -5t^2 + 100}}$

III Integration Formulas

The following is a table of integration formulas with corresponding differentiation formulas studied in lesson 221.20-2.

DIFFERENTIATION FORMULA	CORRESPONDING INTEGRATION FORMULA
$\frac{d}{dx} C = 0$	$\int 0dx = C$
$\frac{d}{dx} x^n = nx^{n-1}$	$\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$
$\frac{d}{dx} (f(x) \pm g(x)) = \frac{d}{dx} f(x) \pm \frac{d}{dx} g(x)$	$\int (f(x) \pm g(x))dx = \int f(x)dx \pm \int g(x)dx$
$\frac{d}{dx} e^{f(x)} = e^{f(x)} f^1(x)$	$\int e^{f(x)} f^1(x)dx = e^{f(x)} + C$

Example 5

$$\int x^{20} dx = \frac{x^{21}}{21} + C$$

Example 6

$$\int \pi x^5 dx = \pi \int x^5 dx$$

$$= \pi (\frac{x^6}{6} + C)$$

$$= \frac{\pi}{6} x^6 + C_1 \qquad (C_1 = \pi C)$$

Example 7

$$\int(x^3 + \sqrt{x})dx = \int x^3 dx + \int\sqrt{x}\,dx$$

$$= \frac{x^4}{4} + C_1 + \frac{x^{3/2}}{3/2} + C_2$$

$$= \frac{1}{4}x^4 + \frac{2}{3}x^{3/2} + C \qquad (C = C_1 + C_2)$$

Example 8

$$\int\frac{x^2 - 1}{\sqrt{x}}\,dx = \int(\frac{x^2}{\sqrt{x}} - \frac{1}{\sqrt{x}})\,dx$$

$$= \int(x^{3/2} - x^{-1/2})dx$$

$$= \int x^{3/2}\,dx - \int x^{-1/2}dx$$

$$= \frac{x^{5/2}}{5/2} + C_1 - (\frac{x^{1/2}}{1/2} + C_2)$$

$$= \frac{2}{5}x^{5/2} + 2\sqrt{x} + C \qquad (C = C_1 - C_2)$$

Note that the integrand in this example was expressed in terms of functions of the form x^n prior to integration, since no method of integrating a quotient of two functions has been given.

Example 9

$$\int 2xe^{x^2}dx = e^{x^2} + C$$

Note that this integral is of the form $\int e^{f(x)}f^1(x)dx$ where $f(x) = x^2$ and $f^1(x) = 2x$.

Example 10

If $v(t) = 10e^{-t} + t$, find $s(t)$ assuming $s(0) = 0$.

Solution

$$s(t) = \int v(t)dt$$

$$= \int (10e^{-t} + t)dt$$

$$= \int 10e^{-t}dt + \int tdt$$

$$= -10 \int e^{-t}(-1)dt + \int tdt$$

$$= -10e^{-t} + \frac{t^2}{2} + C$$

But $s(0) = 0 \Rightarrow -10e^0 + \frac{0^2}{2} + C = 0$

$$\text{ie, } -10 + C = 0$$

$$\text{ie, } \quad C = 10$$

$$\therefore \quad \underline{\underline{s(t) = -10e^{-t} + \frac{t^2}{2} + 10}}$$

Note that "-10" rather than 10 was factored out of the first integrand in line 4 of this solution so as to leave a factor of (-1) in the integrand, which is of the form $e^{f(t)}f^1(t)$ where $f(t) = e^{-t}$ and $f^1(t) = -1$.

IV Area Under a Curve

Let $A(x_1)$ represent the area under the curve $y = f(x)$ from $x = a$ to $x = x_1$. Then $A(x_1 + \Delta x)$ represents the area from $x = a$ to $x = x_1 + \Delta x$, and $A(x_1 + \Delta x) - A(x_1)$ represents the area under the curve between x_1 and $x_1 + \Delta x$, as labelled in Figure 2.

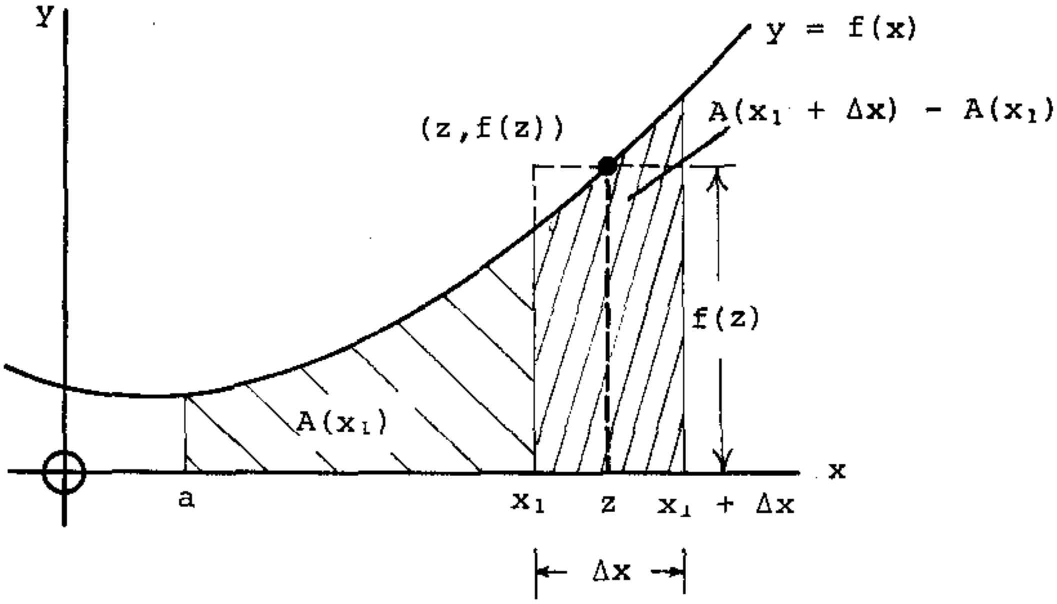

Figure 2

Obviously, for some value of x, say $x = z$, $x_1 \le z \le x_1 + \Delta x$, the area of the rectangle Δx units wide by $f(z)$ units high exactly equals the area $A(x_1 + \Delta x) - A(x_1)$ under the curve,

ie, $$A(x_1 + \Delta x) - A(x_1) = f(z)\Delta x$$

$$\therefore \quad \frac{A(x_1 + \Delta x) - A(x_1)}{\Delta x} = f(z)$$

$$\therefore \quad \lim_{\Delta x \to 0} \frac{A(x_1 + \Delta x) - A(x_1)}{\Delta x} = \lim_{\Delta x \to 0} f(z)$$

The LHS of this equation is the derivative, $A^1(x_1)$, by definition (see lesson 221.20-2).

Furthermore as $\Delta x \to 0$, $z \to x_1$ (see Figure 2).

$$\therefore \quad A^1(x_1) = f(x_1)$$

Finally, since the value of x_1 is arbitrary, it can be replaced by the variable x.

Thus $A^1(x) = f(x)$

ie, the derivative of the area function equals the curve function.

$$\therefore \quad \boxed{A(x) = \int f(x)dx = F(x) + C}$$

Example 11

Find the area under the curve $y = 5$ from $x = 0$ to (a) $x = x_1$ (b) $x = 1$ (c) $x = 10$.

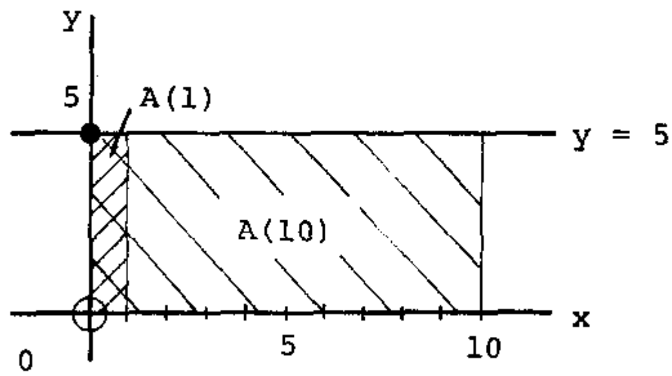

Figure 3

Solution

(a) $A(x) = \int f(x)dx$

$= \int 5dx$

$= 5x + C$

But $A(0) = 0 \Rightarrow 5(0) + C = 0$

$\therefore \quad C = 0$

$\therefore \quad \underline{\underline{A(x_1) = 5x_1}}$

(b) $A(1) = 5(1) = \underline{\underline{5}}$

(c) $A(10) = 5(10) = \underline{\underline{50}}$

Note, with reference to Figure 3, that the above areas are rectangular, and that the same answers are obtained by using "area = length x width".

Example 12

Find the area under the curve $y = 2x$ from $x = 0$ to (a) $x = x_1$ (b) $x = 1$ (c) $x = 10$.

Solution

$$A(x) = \int f(x)\,dx$$
$$= \int 2x\,dx$$
$$= x^2 + C$$

But $A(0) = 0 \Rightarrow 0^2 + C = 0$

$\therefore \quad C = 0$

$\therefore \quad A(x) = x^2$

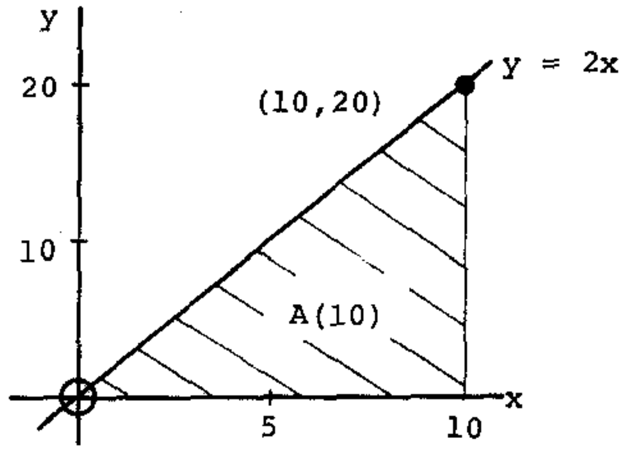

Figure 4

(a) $\therefore \quad \underline{\underline{A(x_1) = x_1^2}}$

(b) $A(1) = 1^2 = \underline{\underline{1}}$

(c) $A(10) = 10^2 = \underline{\underline{100}}$

Note, with reference to Figure 4, that the above areas are triangular, and that the same answers are obtained using "area = $\frac{1}{2}$ base times height".

Example 13

Find the area under the curve $y = x^2$ from $x = 0$ to (a) $x = 5$ (b) $x = 10$.

Solution

$$A(x) = \int x^2\,dx$$
$$= \frac{x^3}{3} + C$$

But $A(0) = 0 \Rightarrow \frac{0^3}{3} + C = 0$

$\therefore \quad C = 0$

$\therefore \quad \underline{\underline{A(x) = \frac{x^3}{3}}}$

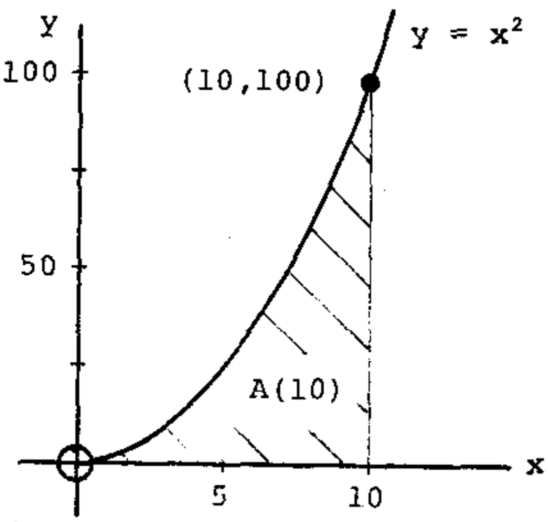

Figure 5

(a) $A(5) = \underline{\underline{\frac{125}{3}}}$

(b) $A(10) = \underline{\underline{\frac{1000}{3}}}$

V The Definite Integral

The procedure for calculating areas illustrated in examples 11 to 13 above can be 'streamlined' considerably by using the definite integral notation, which effectively 'builds in' the boundary condition.

Recall that the area under the curve $y = f(x)$ from $x = a$ up to $x = x$ is given by

$$A(x) = \int f(x)dx = F(x) + C$$

But $A(a) = 0 \Rightarrow F(a) + C = 0$

$$\therefore \quad C = -F(a)$$

$$\therefore \quad A(x) = F(x) - F(a)$$

$\therefore$ the area under the curve between $x = a$ and $x = b$,

$$A(b) = F(b) - F(a)$$

The *definite integral* notation $\int_a^b f(x)dx$ is used to represent $F(b) - F(a)$.

ie,

$$\boxed{A(b) = \int_a^b f(x)dx = F(b) - F(a)}$$

In this notation "a" and "b" are called the *lower limit* and *upper limit*, respectively, of the integration.

Examples 11 and 13 will now be redone using definite integrals:

Example 14

Find the area under the curve $y = 5$ between $x = 0$ and (a) $x = x_1$ (b) $x = 1$ (c) $x = 10$.

Solution

(a) $A(x_1) = \int_0^{x_1} 5dx$

$= 5x\Big|_0^{x_1}$

$= 5x_1 - 5(0)$

$= \underline{\underline{5x_1}}$

The notation $F(x)\Big|_a^b$ is used above to indicate the explicit form of the antiderivative, which is to be evaluated between the limits a and b.

(b) $A(1) = \int_0^{1} 5dx$

$= 5x\Big|_0^{1}$

$= 5(1) - 5(0)$

$= \underline{\underline{5}}$

(c) $A(10) = \int_0^{10} 5dx$

$= 5x\Big|_0^{10}$

$= 5(10) - 5(0)$

$= \underline{\underline{50}}$

Example 15

Find the area under the curve $y = x^2$ from $x = 0$ to (a) $x = 5$ (b) $x = 10$.

Solution

(a) $$A(5) = \int_0^5 x^2 dx$$

$$= \frac{x^3}{3} \Big|_0^5$$

$$= \frac{5^3}{3} - \frac{0^3}{3}$$

$$= \frac{125}{3}$$

(b) $$A(10) = \int_0^{10} x^2 dx$$

$$= \frac{x^3}{3} \Big|_0^{10}$$

$$= \frac{10^3}{3} - \frac{0^3}{3}$$

$$= \frac{1000}{3}$$

ASSIGNMENT

1. Differentiation and integration are opposite processes, yet the derivative is unique whereas the integral is not, in general. Explain why this is so.

2. Why is a boundary condition necessary to obtain a unique solution to an integral? Explain with reference to graphical significance of boundary condition.

3. Evaluate the following indefinite integrals:

   (a) $\int -3x\,dx$ (b) $\int (e^t + t^3)\,dt$

   (c) $\int (2x^2 + 3x - 5)\,dx$ (d) $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

   (e) $\int (4x^3 - \sqrt[3]{x})\,dx$ (f) $\int (2te^{-t^2} + t^{3/2})\,dt$

4. Find displacement and velocity functions s(t) and v(t), respectively, given acceleration function a(t) and boundary conditions as follows:

   (a) $a(t) = 0$, $v(0) = 0$, $s(0) = 0$

   (b) $a(t) = 2$, $v(0) = 10$, $s(0) = 14$

   (c) $a(t) = 2t$, $v(0) = V_0$, $s(0) = 0$

   (d) $a(t) = e^{-t}$, $v(0) = 10$, $s(0) = -10$

5. Find the function which gives the vertical displacement of a projectile relative to ground level, neglecting air resistance. The acceleration due to gravity is 9.8 m/s^2 downwards. Assume the projectile is given initial velocity V_0 m/s vertically upwards from ground level.

6. Find the area under the curve $y = f(x)$ between a and b where

   (a) $f(x) = 2x$, $a = 1$, $b = 10$

   (b) $f(x) = \sqrt{x}$, $a = 4$, $b = 16$

   (c) $f(x) = x^2 + 4$, $a = -2$, $b = 5$

   (d) $f(x) = xe^{-x^2}$, $a = 0$, $b = 2$

7. Evaluate the following definite integrals:

   (a) $\int_1^9 \sqrt{x}\,(x - 1)\,dx$

   (b) $\int_{-5}^0 (-5x + 3)\,dx$

   (c) $\int_0^2 (10 + t - e^t)\,dt$

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